Random Variables and Probability Distributions

If S is a sample space with a probability measure and X is a real value function defined over the elements of S, then X is called a random variable. A random variable X is discrete random variable if its set of possible outcomes is a finite or countable number of values on the real number axis. It is continuous random variable it requires a continuum of values on the real number axis.

Discrete Probability Distribution

If X is a discrete random variable, the function given by $f(x) = P(X = x)$ for each x with in the range of X is called the probability distribution of X. The ordered pair $\{x, f(x)\}$ is called the probability distribution or the probability function of the discrete random variable X.

A function can serve as the probability distribution of a discrete random variable X if and only if, $f(x)$ satisfy the conditions

$$f(x) \geq 0 \qquad \text{for each value/element within its domain}$$

$$\sum_x f(x) = 1$$

where summation extends over all the values in its domain.

Discrete Distribution Function

If X is a discrete random variable, the function given by

$$F(x) = P(X \leq x) = \sum_{t \leq x} f(t) \qquad -\infty < x < \infty$$

where $f(t)$ is the value of probability distribution of X at t. $F(x)$ is called the Distribution Function or the Cumulative Distribution of X. It satisfy the following conditions.

$$F(-\infty) = 0 \quad ; \quad F(+\infty) = 1$$

$$\text{For } a < b \quad ; \quad F(a) \leq F(b) \quad ; \quad a, b \text{ any real numbers}$$

Theorem: If the range of random variable X consists of the values $x_1 < x_2 < x_3 < \cdots x_n$, then $f(x_1) = F(x_1)$ and

$$f(x_i) = F(x_i) - F(x_{i-1})$$

Probability Density Function/Continuous Probability Distribution

A function with values $f(x)$, defined over the set of all real numbers, is called a probability density function of the continuous random variable X if and only if

$$P(a \leq X \leq b) = \int_a^b f(x)\, dx$$

for any real values a and b with $a \leq b$

$$P(X = c) = 0 \qquad \text{for any real constant } c$$

and

$$P(a \leq X \leq b) = P(a \leq X < b) = P(a < X \leq b) = P(a < X < b)$$

A function can act serve as the probability density function of a continuous random variable X if its values, $f(x)$, satisfies the following conditions

$$f(x) \geq 0 \qquad -\infty < x < \infty$$
$$\int_{-\infty}^{+\infty} f(x)\, dx = 1$$

Continuous Distribution Function

If X is a continuous random variable and the value of its probability density function at t is $f(t)$, then the function given by

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\, dt \qquad -\infty < x < +\infty$$

is called distribution function or the cumulative distribution of X.

Theorem: If $f(x)$ and $F(x)$ are the values of the probability density and the distribution function of X at x, then

$$P(a \leq X \leq b) = F(b) - F(a) = \int_{a}^{b} f(x)\, dx$$

for any real constants a and b, with $a \leq b$.

If the cumulative distribution function $F(x)$ is known, probability density is

$$f(x) = \frac{d}{dx} F(x) \qquad \text{; where it exists}$$

Expected Values

The average value of a set of measurements is an example of expectation value. The expected value $E(X)$ is usually referred to as the average value and is given by

$$E(X) = \sum_x x\, f(x) \qquad \text{Discrete random variable}$$
$$E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx \qquad \text{Continuous random variable}$$

In general the expected value of $g(X)$ is

$$E[g(X)] = \sum_x g(x)\, f(x) \qquad \text{Discrete random variable}$$
$$E[g(X)] = \int_{-\infty}^{\infty} g(x)\, f(x)\, dx \qquad \text{Continuous random variable}$$

Theorems

$$E(aX + b) = a\,E(x) + b$$
$$E(aX) = a\,E(X)$$
$$E(b) = b$$
$$E(aX + bY) = a\,E(X) + b\,E(Y)$$
$$E[X \cdot Y] = E(X) \cdot E(Y) \qquad \text{If } X \text{ and } Y \text{ are independent}$$

Mean of Distribution/Moments

The moments about the origin of a probability distribution are the expected values of the random variable that has the given distribution. The rth moment about the origin is

$$\mu'_r = E(X^r) = \sum_x x^r \cdot f(x) \qquad r = 0, 1, 2, \ldots, X\text{Discrete}$$
$$\mu'_r = E(X^r) = \int_{-\infty}^{\infty} x^r \cdot f(x)\,dx \qquad X \text{ is continuous}$$
$$\mu'_1 = E(X) \Rightarrow \text{mean of } X \stackrel{\triangle}{=} \mu$$

Moments about Mean

The rth moment about the mean usually denoted by μ_r is

$$\mu_r = E[(X-\mu)^r] = \sum_x (x-\mu)^r f(x) \qquad X \text{ discrete}$$
$$\mu_r = E[(X-\mu)^r] = \int_{-\infty}^{\infty} (x-\mu)^r f(x)\,dx \qquad X \text{ continuous}$$

This gives

$$\mu_0 = 1 \quad ; \quad \mu_1 = 0$$

The moment μ_2 is called the variance of the random variable X and is denoted by σ^2

$$\mu_2 = \sigma^2 = Var(X) = V(X) \stackrel{\triangle}{=} \text{variance of the distribution}$$

The square root of variance, σ, is called the standard deviation. Moreover

$$\sigma^2 = \mu'_2 - \mu^2 = E[(X-\mu)^2]$$
$$Var(aX + b) = a^2\sigma^2$$