Discrete Probability Distributions

Bernouli Trial/Distribution

A single experiment which has two possible outcomes, usually 0 and 1.

$$f(x) = P(X = x) = \begin{cases} p & \text{if } x = 1 \text{ (true)} \\ 1 - p = q & \text{if } x = 0 \text{ (alse)} \\ 0 & \text{otherwise} \end{cases}$$

or

$$f(x) = P(X = x) = p^x (1-p)^{1-x} \qquad\qquad x = 0, 1$$

Moreover

$$E[X] = \mu = p$$
$$V[X] = \sigma^2 = p(1-p) = \mu_2$$

The distribution of heads and tails in coin tossing is a Bernouli Distribution/Trial with $p = q = 1/2 = 0.5$.

Binomial Distribution

The probability of x successes in n independent Bernouli trials with parameter p of success is

$$f(x) = P(X = x) = \binom{n}{x} p^x (1-p)^{n-x} \qquad ; x = 0, 1, 2, \ldots, n$$

It satisfies the following relation

$$P(X = x, p) = P(X = n - x, (1-p))$$

For example

$$P(X = 11, 0.70) = P(X = 7, 0.30) \qquad\qquad n = 18$$

Moreover

$$\mu = E[X] = np$$
$$V[X] = \sigma^2 = np(1-p) = \mu_2$$

Uniform Distribution

A distribution which has a constant probability is called a uniform distribution.

$$P(X = x) = f(x) = \frac{1}{k} \qquad x = x_1, x_2, x_3, \ldots, x_k$$

where $x_i \neq x_j$ when $i \neq j$

Special Case:

$$\begin{aligned}
f(x) &= P(X = x) = \frac{1}{k} \qquad\qquad x = 1, 2, 3, \ldots, k \\
F(x) &= \frac{x}{k} \\
E[X] &= \mu = \frac{k+1}{2} \\
V[X] &= \sigma^2 = \frac{k^2 - 1}{12} = \mu_2
\end{aligned}$$

It is sometimes also called Rectangular Distribution. The probability distribution for the roll of a die with six side is a uniform distribution with $k = 6, \mu = 3.5, \sigma = 1.7$.

Polya/Pascal (Negative Binomial) Distribution

Let X denotes the number of Bernouli trials till the rth success occurs, where probability of success is p

$$\begin{aligned}
P(X = x) &= \binom{x-1}{r-1} p^r (1-p)^{x-r} \\
E[X] &= \mu = \frac{r}{p} \\
V[X] &= \sigma^2 = \frac{r(1-p)}{p^2}
\end{aligned}$$

If $r = 1$, the distribution is called Geometric.

$$\begin{aligned}
f(x) &= P(X = x) = p(1-p)^{x-1} = pq^{x-1} \\
F(x) &= \sum_{i=1}^{k} P(k) = 1 - (1-p)^x = 1 - q^x \\
E[X] &= \mu = \frac{1}{p} \\
V[X] &= \sigma^2 = \frac{1-p}{p^2} = \frac{q}{p^2} = \mu_2
\end{aligned}$$

The geometric distribution can be used to determine the probability that a given event will occur on a particular trial.

The probability of a rolling a 1 using a single die is $1/6$. If you are rolling the die until you get 1, the probability you will have to roll five times is

$$P(5) = \frac{1}{6}\left(1 - \frac{1}{6}\right)^{5-1} = 0.08$$

Hypergeometric Distribution

The discrete random variable X has a hypergeometric distribution if its probability function is

$$P(X = x) = f(x) = \frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}} \qquad ; x = 0, 1, 2, \ldots, [n, k]$$

where $[n, k]$ means the smaller of two numbers n and k. Moreover

$$E[X] = \mu = \frac{kn}{N}$$

$$V[X] = \frac{k(N-k)n(N-n)}{N^2(N-1)} = \mu_2$$

The distribution arises when elements are drawn from a set without replacement. The random variable X is the number of times that an element of a given type is drawn.

Poisson Distribution

The discrete random variable X has a Poisson distribution if its probability distribution function is

$$f(x) = P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!} \qquad x = 0, 1, 2 \quad \lambda > 0$$

Moreover

$$E[X] = \mu = \lambda$$

$$V[X] = \sigma^2 = \lambda$$

where λ is the rate.

The poisson distribution is the limiting case of the binomial distribution when $n \to \infty$ and $x \to 0$ such that $nx \to \lambda$.