Continuous Probability Distributions

Exponential Distribution

The continuous random variable x has an exponential distribution if its probability distribution function $f(x)$ is given by the equation below. The parameter λ is called rate of change. Defining the mean waiting time between successive changes as $\theta = \lambda^{-1} = \frac{1}{\lambda}$, we get

$$f(x) = \lambda e^{-\lambda x} = \frac{1}{\theta} e^{-x/\theta} \qquad x \geq 0$$

$$F(x) = \begin{cases} 1 - e^{-\lambda x} = 1 - e^{-x/\theta} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

The mean and variance is given by

$$E[X] = \mu = \frac{1}{\lambda} = \theta$$

$$V[X] = \sigma^2 = \frac{1}{\lambda^2} = \theta^2$$

The exponential distribution is often used to describe the amount of radioactive substance remaining at a given time.

Uniform Distribution

A distribution which has constant probability is called a uniform distribution, sometimes also called a rectangular distribution. The probability density and cumulative distribution functions are

$$f(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{Elsewhere} \end{cases}$$

$$F(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \leq x \leq b \\ 1 & x > b \end{cases}$$

The mean and variance of the uniform distribution are

$$E[X] = \mu = \frac{a+b}{2}$$

$$V[X] = \sigma^2 = \frac{(b-a)^2}{12} = \mu_2$$

Gaussian/Normal Distribution

A distribution with mean μ and standard deviation σ is normal if

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} \qquad -\infty < x < \infty$$

$$F(x) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{x} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}\, dx$$

$$E[X] = \mu$$

$$V[X] = \sigma^2$$

The short notation is

$$x \sim N(\mu, \sigma^2)$$

The normal distribution is useful for describing very large sample groups. According to central limit theorem, many probability distributions approach the normal distribution as the number of elements in sample group approaches infinity.

Standard Normal Distribution

The normal distribution with $\mu = 0$ and $\sigma^2 = 1 = 0$ is referred as standard normal distribution.

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{\frac{-z^2}{2}} \qquad -\infty < z < \infty$$

$$F(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{\frac{-z^2}{2}}\, dz$$

$$E[Z] = \mu = 0$$

$$V[Z] = \sigma^2 = 1 = \mu_2$$

The short notation is as under

$$Z \sim N(0, 1)$$

Theorem: If X has a normal distribution with the mean μ and the standard deviation σ, then

$$Z = \frac{X - \mu}{\sigma} \Rightarrow dz = \frac{dx}{\sigma}$$

has the standard normal distribution for the transformed distribution. The z defined in this way is known as $z-$score.