Trigonometry

Reduction Formula

$$f(\pm\alpha + n90^\circ) = \pm g(\alpha)$$

Where

$$\begin{aligned} n &= \text{any integer, positive, negative or zero} \\ f &= \text{any one of the six trigonometric functions} \\ \alpha &= \text{any real angle measure} \end{aligned}$$

- If n is even, then g is the same function as f.
- If n is odd, then g is the co-function of f. (sin and cos, tan and cot, and sec and csc are co-functions of each other.)

The second $\pm$ is determined by the quadrant in which angle $(\pm\alpha + 90^\circ)$ lies.

Fundamental Identities

Reciprocal Relations

$$\sin\alpha = \frac{1}{\csc\alpha} \qquad \cos\alpha = \frac{1}{\sec\alpha} \qquad \tan\alpha = \frac{1}{\cot\alpha}$$
$$\csc\alpha = \frac{1}{\sin\alpha} \qquad \sec\alpha = \frac{1}{\cos\alpha} \qquad \cot\alpha = \frac{1}{\tan\alpha}$$

Product Relations

$$\begin{aligned} \sin\alpha &= \tan\alpha\cos\alpha & \cos\alpha &= \cot\alpha\sin\alpha \\ \tan\alpha &= \sin\alpha\sec\alpha & \cot\alpha &= \cos\alpha\csc\alpha \\ \sec\alpha &= \csc\alpha\tan\alpha & \csc\alpha &= \sec\alpha\cot\alpha \end{aligned}$$

Quotient Relations

$$\sin\alpha = \frac{\tan\alpha}{\sec\alpha} \qquad \cos\alpha = \frac{\cot\alpha}{\csc\alpha} \qquad \tan\alpha = \frac{\sin\alpha}{\cos\alpha}$$
$$\csc\alpha = \frac{\sec\alpha}{\tan\alpha} \qquad \sec\alpha = \frac{\csc\alpha}{\cot\alpha} \qquad \cot\alpha = \frac{\cos\alpha}{\sin\alpha}$$

Pythagorean Relations

$$\begin{aligned} \sin^2\alpha + \cos^2\alpha &= 1 \\ 1 + \tan^2\alpha &= \sec^2\alpha \\ 1 + \cot^2\alpha &= \csc^2\alpha \end{aligned}$$

Angle-sum and Angle-difference Relations

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$
$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$
$$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$
$$\cot(\alpha+\beta) = \frac{\cot\beta\cot\alpha - 1}{\cot\beta + \cot\alpha}$$
$$\cot(\alpha-\beta) = \frac{\cot\beta\cot\alpha + 1}{\cot\beta - \cot\alpha}$$
$$\sin(\alpha+\beta)\sin(\alpha-\beta) = \sin^2\alpha - \sin^2\beta = \cos^2\beta - \cos^2\alpha$$
$$\cos(\alpha+\beta)\cos(\alpha-\beta) = \cos^2\alpha - \sin^2\beta = \cos^2\beta - \sin^2\alpha$$

Double-angle Relations

$$\sin 2\alpha = 2\sin\alpha\cos\alpha = \frac{2\tan\alpha}{1+\tan^2\alpha}$$
$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha = \frac{1-\tan^2\alpha}{1+\tan^2\alpha}$$
$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha} \qquad , \qquad \cot 2\alpha = \frac{\cot^2\alpha - 1}{2\cot\alpha}$$

Multiple-angle Relations

$$\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$$
$$\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$$
$$\sin 4\alpha = 4\sin\alpha\cos\alpha - 8\sin^3\alpha\cos\alpha$$
$$\cos 4\alpha = 8\cos^4\alpha - 8\cos^2\alpha + 1$$
$$\sin n\alpha = 2\sin(n-1)\alpha\cos\alpha - \sin(n-2)\alpha$$
$$\cos n\alpha = 2\cos(n-1)\alpha\cos\alpha - \cos(n-2)\alpha$$
$$\tan 3\alpha = \frac{3\tan\alpha - \tan^3\alpha}{1 - 3\tan^2\alpha}$$
$$\tan 4\alpha = \frac{4\tan\alpha - 4\tan^3\alpha}{1 - 6\tan^2\alpha + \tan^4\alpha}$$
$$\tan n\alpha = \frac{\tan(n-1)\alpha + \tan\alpha}{1 - \tan(n-1)\alpha\tan\alpha}$$

Function-product Relations

$$\sin\alpha\sin\beta = \frac{1}{2}\cos(\alpha-\beta) - \frac{1}{2}\cos(\alpha+\beta)$$
$$\cos\alpha\cos\beta = \frac{1}{2}\cos(\alpha-\beta) + \frac{1}{2}\cos(\alpha+\beta)$$
$$\sin\alpha\cos\beta = \frac{1}{2}\sin(\alpha+\beta) + \frac{1}{2}\sin(\alpha-\beta)$$
$$\cos\alpha\sin\beta = \frac{1}{2}\sin(\alpha+\beta) - \frac{1}{2}\sin(\alpha-\beta)$$

Function-sum and Function-difference Relations

$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$
$$\sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$
$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$
$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$
$$\tan\alpha + \tan\beta = \frac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta} \quad , \quad \tan\alpha - \tan\beta = \frac{\sin(\alpha-\beta)}{\cos\alpha\cos\beta}$$
$$\cot\alpha + \cot\beta = \frac{\sin(\alpha+\beta)}{\sin\alpha\sin\beta} \quad , \quad \cot\alpha - \cot\beta = \frac{\sin(\beta-\alpha)}{\sin\alpha\sin\beta}$$

Half-angle Relations

$$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}} \quad , \quad \cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}$$
$$\tan\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} = \frac{1-\cos\alpha}{\sin\alpha} = \frac{\sin\alpha}{1+\cos\alpha}$$
$$\cot\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{1-\cos\alpha}} = \frac{1+\cos\alpha}{\sin\alpha} = \frac{\sin\alpha}{1-\cos\alpha}$$

Power Relations

$$\sin^2\alpha = \frac{1-\cos 2\alpha}{2} \quad , \quad \sin^3\alpha = \frac{3\sin\alpha - \sin 3\alpha}{4}$$
$$\sin^4\alpha = \frac{3-4\cos 2\alpha + \cos 4\alpha}{8}$$
$$\cos^2\alpha = \frac{1+\cos 2\alpha}{2} \quad , \quad \cos^3\alpha = \frac{3\cos\alpha + \cos 3\alpha}{4}$$
$$\cos^4\alpha = \frac{3+4\cos 2\alpha + \cos 4\alpha}{8}$$
$$\tan^2\alpha = \frac{1-\cos 2\alpha}{1+\cos 2\alpha} \quad , \quad \cot^2\alpha = \frac{1+\cos 2\alpha}{1-\cos 2\alpha}$$

Principle Values of the Inverse Trigonometric Functions

$$
\begin{array}{rl}
-\pi/2 \le \mathrm{Sin}^{-1}x \le \pi/2 & -1 \le x \le 1 \\
0 \le \mathrm{Cos}^{-1}x \le \pi & -1 \le x \le 1 \\
-\pi/2 < \mathrm{Tan}^{-1}x < \pi/2 & -\infty < x < \infty \\
0 < \mathrm{Csc}^{-1}x \le \pi/2 & x \ge 1 \\
-\pi < \mathrm{Csc}^{-1}x \le -\pi/2 & x \le -1 \\
0 \le \mathrm{Sec}^{-1}x < \pi/2 & x \ge 1 \\
-\pi \le \mathrm{Sec}^{-1}x < -\pi/2 & x \le -1 \\
0 < \mathrm{Cot}^{-1}x < \pi & -\infty < x < \infty
\end{array}
$$

Plane Triangle Formulae

In the following α, β, γ denotes the angles of any plane triangle; a, b, c the corresponding opposite sides, and $s = \frac{a+b+c}{2}$. The symbol $\triangle$ represents area of the triangle.

Radius of Inscribed Circle

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}} = \frac{\triangle}{s}$$

Radius of Circumscribed Circle

$$R = \frac{a}{2\sin\alpha} = \frac{b}{2\sin\beta} = \frac{c}{2\sin\gamma} = \frac{abc}{4\triangle}$$

Law of Sines

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

Law of Cosines

$$
\begin{array}{ll}
a^2 = b^2 + c^2 - 2bc\cos\alpha & \cos\alpha = \dfrac{b^2+c^2-a^2}{2bc} \\
b^2 = c^2 + a^2 - 2ca\cos\beta & \cos\beta = \dfrac{c^2+a^2-b^2}{2ca} \\
c^2 = a^2 + b^2 - 2ab\cos\gamma & \cos\gamma = \dfrac{a^2+b^2-c^2}{2ab}
\end{array}
$$

Law of Tangents

$$\frac{b-c}{b+c} = \frac{\tan\frac{\beta-\gamma}{2}}{\tan\frac{\beta+\gamma}{2}} \quad , \quad \frac{c-a}{c+a} = \frac{\tan\frac{\gamma-\alpha}{2}}{\tan\frac{\gamma+\alpha}{2}}$$

$$\frac{a-b}{a+b} = \frac{\tan\frac{\alpha-\beta}{2}}{\tan\frac{\alpha+\beta}{2}}$$

Half-angle Formulae

$$\tan\frac{\alpha}{2} = \frac{r}{s-a} \quad , \quad \tan\frac{\beta}{2} = \frac{r}{s-b} \quad , \quad \tan\frac{\gamma}{2} = \frac{r}{s-c}$$

$$\sin\frac{\alpha}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}} \qquad \cos\frac{\alpha}{2} = \sqrt{\frac{s(s-a)}{bc}}$$

$$\sin\frac{\beta}{2} = \sqrt{\frac{(s-a)(s-c)}{ac}} \qquad \cos\frac{\beta}{2} = \sqrt{\frac{s(s-b)}{ac}}$$

$$\sin\frac{\gamma}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}} \qquad \cos\frac{\gamma}{2} = \sqrt{\frac{s(s-c)}{ab}}$$

Area

$$\triangle = \tfrac{1}{2}bc\sin\alpha = \tfrac{1}{2}ca\sin\beta = \tfrac{1}{2}ab\sin\gamma$$

$$\triangle = \frac{a^2\sin\beta\sin\gamma}{2\sin\alpha} = \frac{b^2\sin\gamma\sin\alpha}{2\sin\beta} = \frac{c^2\sin\alpha\sin\beta}{2\sin\gamma}$$

$$\triangle = \sqrt{s(s-a)(s-b)(s-c)} = rs = \frac{abc}{4R}$$

Mollweide's Formulae

$$\frac{b-c}{a} = \frac{\sin\frac{\beta-\gamma}{2}}{\cos\frac{\alpha}{2}} \quad , \quad \frac{c-a}{b} = \frac{\sin\frac{\gamma-\alpha}{2}}{\cos\frac{\beta}{2}}$$

$$\frac{a-b}{c} = \frac{\sin\frac{\alpha-\beta}{2}}{\cos\frac{\gamma}{2}}$$

Newton's Formulae

$$\frac{b+c}{a} = \frac{\cos\frac{\beta-\gamma}{2}}{\sin\frac{\alpha}{2}} \quad , \quad \frac{c+a}{b} = \frac{\cos\frac{\gamma-\alpha}{2}}{\sin\frac{\beta}{2}}$$

$$\frac{a+b}{c} = \frac{\cos\frac{\alpha-\beta}{2}}{\sin\frac{\gamma}{2}}$$

Hyperbolic Trigonometry

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{csch} x = \frac{1}{\sinh x}$$

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{sech} x = \frac{1}{\cosh x}$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} \qquad \coth x = \frac{1}{\tanh x}$$

Fundamental Identities

$$\cosh x + \sinh x = e^{x}$$
$$\cosh x - \sinh x = e^{-x}$$

$$\sinh(-x) = -\sinh x \qquad \operatorname{csch}(-x) = -\operatorname{csch} x$$
$$\cosh(-x) = \cosh x \qquad \operatorname{sech}(-x) = \operatorname{sech} x$$
$$\tanh(-x) = -\tanh x \qquad \coth(-x) = -\coth x$$

$$\cosh^2 x - \sinh^2 x = 1 \qquad \tanh^2 x + \operatorname{sech}^2 x = 1$$
$$\coth^2 x - \operatorname{csch}^2 x = 1 \qquad \operatorname{csch}^2 x - \operatorname{sech}^2 x = \operatorname{csch}^2 x \operatorname{sech}^2 x$$

$$\sinh(u+v) = \sinh u \cosh v + \cosh u \sinh v$$
$$\sinh(u-v) = \sinh u \cosh v - \cosh u \sinh v$$
$$\cosh(u+v) = \cosh u \cosh v + \sinh u \sinh v$$
$$\cosh(u-v) = \cosh u \cosh v - \sinh u \sinh v$$
$$\tanh(u+v) = \frac{\tanh u + \tanh v}{1 + \tanh u \tanh v}$$
$$\tanh(u-v) = \frac{\tanh u - \tanh v}{1 - \tanh u \tanh v}$$
$$\sinh(u+v)\sinh(u-v) = \sinh^2 u - \sinh^2 v = \cosh^2 u - \cosh^2 v$$
$$\cosh(u+v)\cosh(u-v) = \sinh^2 u + \cosh^2 v = \cosh^2 u + \sinh^2 v$$
$$\sinh u \cosh v = \tfrac{1}{2}\sinh(u+v) + \tfrac{1}{2}\sinh(u-v)$$
$$\cosh u \sinh v = \tfrac{1}{2}\sinh(u+v) - \tfrac{1}{2}\sinh(u-v)$$
$$\cosh u \cosh v = \tfrac{1}{2}\cosh(u+v) + \tfrac{1}{2}\cosh(u-v)$$
$$\sinh u \sinh v = \tfrac{1}{2}\cosh(u+v) - \tfrac{1}{2}\cosh(u-v)$$
$$\sinh u + \sinh v = 2\sinh\tfrac{u+v}{2}\cosh\tfrac{u-v}{2}$$
$$\sinh u - \sinh v = 2\cosh\tfrac{u+v}{2}\sinh\tfrac{u-v}{2}$$
$$\cosh u + \cosh v = 2\cosh\tfrac{u+v}{2}\cosh\tfrac{u-v}{2}$$
$$\cosh u - \cosh v = 2\sinh\tfrac{u+v}{2}\sinh\tfrac{u-v}{2}$$
$$\sinh x = \frac{2\tanh\frac{x}{2}}{1-\tanh^2\frac{x}{2}} = \frac{\tanh x}{\sqrt{1-\tanh^2 x}}$$
$$\sinh x = \frac{1+\tanh^2\frac{x}{2}}{1-\tanh^2\frac{x}{2}} = \frac{1}{\sqrt{1-\tanh^2 x}}$$
$$\sinh x + \cosh x = \frac{1+\tanh\frac{x}{2}}{1-\tanh\frac{x}{2}}$$

$$\tanh u + \tanh v = \frac{\sinh(u+v)}{\cosh u \cosh v}$$

$$\tanh u - \tanh v = \frac{\sinh(u-v)}{\cosh u \cosh v}$$

$$\coth u + \coth v = \frac{\sinh(u+v)}{\sinh u \sinh v}$$

$$\coth u - \coth v = \frac{\sinh(v-u)}{\sinh u \sinh v}$$

$$\sinh 2x = 2 \sinh x \cosh x$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x$$

$$\tanh 2x = \frac{2\tanh x}{1 + \tanh^2 x}$$

$$\sinh 3x = 3\sinh x + 4\sinh^3 x$$

$$\cosh 3x = 4\cosh^3 x - 3\cosh x$$

$$\tanh 3x = \frac{3\tanh x + \tanh^3 x}{1 + 3\tanh^2 x}$$

$$\sinh \frac{x}{2} = \pm\sqrt{\frac{\cosh x - 1}{2}}$$

$$\cosh \frac{x}{2} = \sqrt{\frac{\cosh x + 1}{2}}$$

$$\tanh \frac{x}{2} = \frac{\cosh x - 1}{\sinh x} = \frac{\sinh x}{\cosh x + 1}$$

$$\cosh^2 x = \frac{\cosh 2x + 1}{2}$$

$$\sinh^2 x = \frac{\cosh 2x - 1}{2}$$

Inverse Hyperbolic Functions

$$\sinh^{-1} x = \ln(x + \sqrt{x^2+1}) \qquad -\infty < x < \infty$$

$$\cosh^{-1} x = \ln(x \pm \sqrt{x^2-1}) \quad x \geq 1; \quad + \text{ sign is for principle value}$$

$$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \qquad |x| < 1$$

$$\operatorname{csch}^{-1} x = \ln\left(\frac{1 \pm \sqrt{1+x^2}}{x}\right) \quad + \text{ sign if } x > 0; \text{ - sign if } x < 0; \ x \neq 0$$

$$\operatorname{sech}^{-1} x = \ln\left(\frac{1 \pm \sqrt{1-x^2}}{x}\right) \quad 0 < x \leq 1; + \text{ sign is for principle value}$$

$$\coth^{-1} x = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right) \qquad |x| > 1$$