Complex Numbers and Variables

Let $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2, z_3 = x_3 + iy_3; z = x + iy$ Where

$$i = \sqrt{-1}$$

Basic Theorems

1. $z_1 = z_2 \qquad$ provided $x_1 = x_2$ and $y_1 = y_2$
2. $z_1 \pm z_2 = (x_1 \pm x_2) + i(y_1 \pm y_2)$
3. $z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$
4. $z^{-1} = \frac{1}{z} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$
5. $\frac{z_1}{z_2} = z_1 z_2^{-1} = \frac{x_1x_2+y_1y_2}{x_2^2+y_2^2} + \frac{x_2y_1-x_1y_2}{x_2^2+y_2^2}$
6. $\overline{z} = x - iy$
7. $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$
8. $\overline{z_1 \pm z_2} = \overline{z_1} \pm \overline{z_2}; \quad \overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}; \quad \overline{z_1/z_2} = \overline{z_1}/\overline{z_2}$
9. $\overline{\overline{z}} = z$
10. $z\overline{z} = x^2 + y^2 = (\Re z)^2 + (\Im z)^2$
11. $\Re z = x = \frac{z+\overline{z}}{2}; \quad \Im z = y = \frac{z-\overline{z}}{2i}$

Various Notations of Complex Numbers

$$z = r(\cos\theta + i\sin\theta) = r\,\mathrm{cis}\,\theta = re^{i\theta} = r\underline{/\theta}$$

Where $|z| = r = \sqrt{x^2+y^2} = \sqrt{z\overline{z}}$

$$\theta = \arg z = \tan^{-1}\frac{y}{x} \qquad -\pi < \theta \le \pi$$

The multiplication and division is defined as under:

$$z_1 z_2 = r_1 r_2\,\mathrm{cis}(\theta_1+\theta_2) = r_1 r_2 e^{i(\theta_1+\theta_2) = r_1 r_2}\underline{/\theta_1+\theta_2}$$
$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\,\mathrm{cis}(\theta_1-\theta_2) = \frac{r_1}{r_2}e^{i(\theta_1-\theta_2)} = \frac{r_1}{r_2}\underline{/\theta_1-\theta_2}$$

Deductions

1. $e^{i\theta} = \cos\theta + i\sin\theta \quad ; \quad e^{-i\theta} = \cos\theta - i\sin\theta$
2. $e^z = e^x \cdot e^{iy} = e^x(\cos y + i\sin y) \quad , \quad |e^z| = e^x$
3. $|z| = |-z| = |\overline{z}|$

4. $|z_1 - z_2| = |z_2 - z_1|$
5. $|z|^2 = \left|z^2\right| = z\overline{z}$
6. $|z_1 z_2| = |z_1|\,|z_2|$
7. $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$
8. $|z_1 + z_2| \leq |z_1| + |z_2|$
9. $||z_1| - |z_2|| \leq |z_1 - z_2|$
10. $|z_1| - |z_2| \leq |z_1 + z_2|$

Demoivre's Theorem, its Deductions

$$z^n = r^n(\cos n\theta + i \sin n\theta) = r^n \operatorname{cis} n\theta = r^n e^{in\theta} = r^n \underline{/n\theta}$$

Complex Roots:

$$\begin{aligned}(\rho \operatorname{cis} \theta)^{\frac{1}{n}} &= (\rho e^{i\theta})^{\frac{1}{n}} = (\rho \underline{/\theta})^{\frac{1}{n}} = \rho^{\frac{1}{n}} \operatorname{cis} \frac{\theta + 2k\pi}{n} \\ &= \rho^{\frac{1}{n}} e^{i\frac{\theta+2k\pi}{n}} = \rho^{\frac{1}{n}} \underline{/\tfrac{\theta+2k\pi}{n}} \qquad k = 0, 1, 2, 3, \cdots, n-1\end{aligned}$$

Complex Variables

$$\begin{aligned} e^{i\theta} &= \cos\theta + i\sin\theta \quad , \quad i = \sqrt{-1} \\ \sin z &= \frac{e^{iz} - e^{-iz}}{2i} \quad , \quad \cos z = \frac{e^{iz} + e^{-iz}}{2} \\ \tan z &= -i\left(\frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}}\right) = -i\left(\frac{e^{2iz} - 1}{e^{2iz} + 1}\right) \\ \ln z &= \ln r + i\theta = ln\,|z| + i \arg\theta \\ \operatorname{Ln} z &= \ln|z| + i \operatorname{Arg} z \\ \ln z &= \operatorname{Ln} z \pm 2n\pi i \\ a^z &= e^{z \ln a} \end{aligned}$$

Relations between Hyperbolic and Trigonometric Functions

$$\begin{aligned} \sinh iz &= i \sin z & \sin iz &= i \sinh z \\ \cosh iz &= \cos z & \cos iz &= \cosh z \\ \tanh iz &= i \tan z & \tan iz &= i \tanh z \end{aligned}$$

$$\begin{aligned} \sinh(u \pm iv) &= \sinh u \cos v \pm i \cosh u \sin v \\ \cosh(u \pm iv) &= \cosh u \cos v \pm i \sinh u \sin v \\ \tanh(u \pm iv) &= \frac{\sinh 2u \pm i \sin 2v}{\cosh 2u + \cos 2v} \\ \coth(u \pm iv) &= \frac{\sinh 2u \mp i \sin 2v}{\cosh 2u - \cos 2v} \end{aligned}$$

$$\sinh(u + \tfrac{\pi}{2}i) = i\cosh u \qquad \cosh(u + \tfrac{\pi}{2}i) = i\sinh u$$
$$\sinh(u + \pi i) = -\sinh u \qquad \cosh(u + \pi i) = -\cosh u$$
$$\sinh(u + 2\pi i) = \sinh u \qquad \cosh(u + 2\pi i) = \cosh u$$