Facts from Algebra

Factors and Expansions

$$(a \pm b)^2 = a^2 \pm 2ab + b^2$$
$$(a \pm b)^3 = a^3 \pm 3a^2b + 3ab^2 \pm b^3$$
$$(a \pm b)^4 = a^4 \pm 4a^3b + 6a^2b^2 \pm 4ab^3 + b^4$$
$$a^2 - b^2 = (a-b)(a+b)$$
$$a^2 + b^2 = (a + b\sqrt{-1})(a - b\sqrt{-1})$$
$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$
$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$
$$a^4 + b^4 = (a^2 + ab\sqrt{2} + b^2)(a^2 - ab\sqrt{2} + b^2)$$
$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + b^{n-1})$$
$$a^n - b^n = (a+b)(a^{n-1} - a^{n-2}b + \cdots - b^{n-1}) \qquad \text{for even } n$$
$$a^n + b^n = (a+b)(a^{n-1} - a^{n-2}b + \cdots + b^{n-1}) \qquad \text{for odd } n$$
$$a^4 + a^2b^2 + b^4 = (a^2 + ab + b^2)(a^2 - ab + b^2)$$
$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$$
$$(a+b+c)^3 = a^3 + b^3 + c^3 + 3a^2(b+c) + 3b^2(a+c) + 3c^2(a+b) + 6abc$$
$$(a+b+c+d+\cdots)^2 = a^2 + b^2 + c^2 + d^2 + \cdots + 2a(b+c+d+\cdots) + 2b(c+d+\cdots) + 2c(d+\cdots) + \cdots$$

Powers and Roots

$$a^x \times a^y = a^{x+y} \qquad a^0 = 1 \quad (a \neq 0) \qquad (ab)^x = a^x b^x$$
$$\frac{a^x}{b^x} = a^{x-y} \qquad a^{-x} = \frac{1}{a^x} \qquad \left(\frac{a}{b}\right) = \frac{a^x}{b^x}$$
$$(a^x)^y = a^{xy} \qquad a^{1/x} = \sqrt[x]{a} \qquad \sqrt[x]{ab} = \sqrt[x]{a}\,\sqrt[x]{b}$$
$$\sqrt[x]{\sqrt[y]{a}} = \sqrt[xy]{a} \qquad a^{x/y} = \sqrt[y]{a^x} \qquad \sqrt[x]{\frac{a}{b}} = \frac{\sqrt[x]{a}}{\sqrt[x]{b}}$$

Laws of Logarithms

$$\log(xy) = \log x + \log y$$
$$\log\left(\frac{x}{y}\right) = \log x - \log y$$
$$\log(x^n) = n \log x$$

Change of Base

$$\log_a x = \frac{\log_b x}{\log_b a} = (\log_b x) \cdot (\log_a b)$$
$$\log_{10} x = 0.4342944819 \log_e x$$
$$\log_e x = 2.3025850930 \log_{10} x$$

Proportions

$$\text{If} \qquad \frac{a}{b} = \frac{c}{d}$$

Then

$$\frac{a+b}{b} = \frac{c+d}{d} \qquad , \qquad \frac{a-b}{b} = \frac{c-d}{d}$$

$$\frac{a-b}{a+b} = \frac{c-d}{c+d}$$

Arithmetic Progression

An arithmetic progression is a sequence of numbers such that each number differs from the previous number by a constant amount called the "common difference".

If a_1 is the first term; a_n the nth term, d the common difference; and s_n the sum of n terms

$$a_n = a_1 + (n-1)d$$

$$s_n = \frac{n}{2}(a_1 + a_n) = \frac{n}{2}\left[2a_1 + (n-1)d\right]$$

The arithmetic mean between a and b is given by $\frac{a+b}{2}$.

Geometric Progression

An geometric progression is a sequence of numbers such that each number bears a constant ratio, called the "common difference", to the previous number.

If a_1 is the first term; a_n the nth term, r the common ratio; n the number of terms and s_n the sum of n terms

$$a_n = a_1 r^{n-1}$$

$$s_n = a_1 \frac{1-r^n}{1-r} = a_1 \frac{r^n - 1}{r-1} \qquad (r \neq 1)$$

$$s_\infty = \lim_{n\to\infty} a_1 \frac{1-r^n}{1-r} = \frac{a_1}{1-r} \qquad (|r| < 1)$$

The geometric mean between a and b is given by $\sqrt{ab}$.

Note:- In case of geometric and arithmetic progression it is customary to represent a_n by l in a finite progression and refer to it as the last term.

Harmonic Progression

A sequence of numbers whose reciprocals form an arithmetic progression is called an harmonic progression. Thus

$$\frac{1}{a_1}, \frac{1}{a_1+d}, \frac{1}{a_1+2d}, \dots, \frac{1}{a_1+(n-1)d}$$

$$\text{where} \quad \frac{1}{a_n} = \frac{1}{a_1+(n-1)d}$$

forms an harmonic progression.
The harmonic mean between a and b is given by $\frac{2ab}{a+b}$.

If A, G, H respectively represent the arithmetic, geometric and harmonic mean between a and b, then

$$G^2 = AH$$

Permutations

If $M = {}_nP_r = P_{n:r}$ denotes the number of permutations of n distinct things taken r at a time, then

$$M = n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}$$

$$\text{where} \quad n! = n(n-1)(n-2)\cdots 1$$

Combinations

If $M = {}_nC_r = C_{n:r} = \binom{n}{r}$ denotes the number of combinations of n distinct things taken r at a time, then

$$M = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!} = \frac{n!}{r!(n-r)!}$$

$$\text{By defination} \quad \binom{n}{0} = 1$$

Quadratic Equations

Any quadratic equation may be reduced to the form

$$ax^2 + bx + c = 0$$

Then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If a, b and c are real then:

- If $b^2 - 4ac > 0$, the roots are real and unequal.
- If $b^2 - 4ac = 0$, the roots are real and equal.
- If $b^2 - 4ac < 0$, the roots are imaginary and unequal (conjugate pair).

Theorems Basic to the Algebra of Sets

1. $\mathbf{A} \cup \mathbf{B} = \mathbf{B} \cup \mathbf{A} \quad ; \quad \mathbf{A} \cap \mathbf{B} = \mathbf{B} \cap \mathbf{A}$
2. $(\mathbf{A} \cup \mathbf{B}) \cup \mathbf{C} = \mathbf{A} \cup (\mathbf{B} \cup \mathbf{C}) \quad ; \quad (\mathbf{A} \cap \mathbf{B}) \cap \mathbf{C} = \mathbf{A} \cap (\mathbf{B} \cap \mathbf{C})$
3. $\mathbf{A} \cap (\mathbf{B} \cup \mathbf{C}) = (\mathbf{A} \cap \mathbf{B}) \cup (\mathbf{A} \cap \mathbf{C}) \quad ; \quad \mathbf{A} \cup (\mathbf{B} \cap \mathbf{C}) = (\mathbf{A} \cup \mathbf{B}) \cap (\mathbf{A} \cup \mathbf{C})$
4. $\mathbf{A} \cup \mathbf{A} = \mathbf{A} \cap \mathbf{A} = \mathbf{A}$
5. $\mathbf{A} \cap \mathbf{U} = \mathbf{A} \cup \emptyset = \mathbf{A} \quad ; \quad \mathbf{A} \cup \mathbf{U} = \mathbf{U} \quad ; \quad \mathbf{A} \cap \emptyset = \emptyset$
6. $(\mathbf{A} \cap \mathbf{B}) \cup (\mathbf{A} \backslash \mathbf{B}) = \mathbf{A}$
7. $(\mathbf{A} \backslash \mathbf{B}) \cup \mathbf{B} = \mathbf{A} \cup \mathbf{B}$
8. $\mathbf{A} \subseteq \mathbf{A} \cup \mathbf{B}$
9. $\mathbf{A} \cap \mathbf{B} \subseteq \mathbf{A}$
10. $\mathbf{A} \cup \mathbf{B} = \mathbf{A}$ iff $\mathbf{B} \subseteq \mathbf{A}$
11. $\mathbf{A} \cap \mathbf{B} = \mathbf{A}$ iff $\mathbf{A} \subseteq \mathbf{B}$
12. $\mathbf{A} \backslash \mathbf{B} = \mathbf{A} \backslash (\mathbf{A} \cap \mathbf{B})$
13. $(\mathbf{A} \backslash \mathbf{B}) \cap (\mathbf{A} \backslash \mathbf{C}) = \mathbf{A} \backslash (\mathbf{B} \cup \mathbf{C}) \quad ; \quad (\mathbf{A} \backslash \mathbf{B}) \cup (\mathbf{A} \backslash \mathbf{C}) = \mathbf{A} \backslash (\mathbf{B} \cap \mathbf{C})$
14. $(\mathbf{A} \cup \mathbf{B})' = \mathbf{A}' \cap \mathbf{B}' \quad ; \quad (\mathbf{A} \cap \mathbf{B})' = \mathbf{A}' \cup \mathbf{B}'$
15. $\mathbf{A} \cup \mathbf{A}' = \mathbf{U} \quad ; \quad \mathbf{A} \cap \mathbf{A}' = \emptyset$