

Spherical Trigonometry and Navigational Calculations

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Abstract—This paper discusses the fundamental concepts in spherical trigonometry and its application to the navigational calculation. It briefly discusses the history of the spherical trigonometry. It describes the commonly used terms in navigation and spherical trigonometry. It specifically concentrates on great-circle and dead-reckoning navigation. It also outlines the various functions available in the Mapping Toolbox of MATLAB for such navigational calculations.

Index Terms— Dead-reckoning, Geodesic, Great-circle, Navigation, Spherical Trigonometry.

I. INTRODUCTION

NAVIGATION is the process of planning, recording, and controlling the movement of a craft or vehicle from one location to another. The word derives from the Latin roots *navis* (“ship”) and *agere* (“to move or direct”). To achieve these goals in a general way, a coordinate system is needed that allow quantitative calculations. The most commonly used notation involves latitudes and longitudes in a spherical coordinate system. *Spherical Trigonometry* deals with triangles drawn on a sphere. The development of spherical trigonometry lead to improvements in the art of earth-surfaced, orbital, space and inertial navigation, map making, positions of sunrise and sunset, and astronomy.

II. HISTORY

Spherical trigonometry was dealt with by early Greek mathematicians such as Menelaus of Alexandria who wrote a book that dealt with spherical trigonometry called “*Sphaerica*” [1]. The subject further developed in the Islamic Caliphates of the Middle East, North Africa and Spain during the 8th to 14th centuries. It arose to solve an apparently simple problem: Which direction is Mecca [2]? In the 10th century, Abu al-Wafa al-Buzjani established the angle addition identities, e.g. $\sin(a + b)$, and discovered the law of sines for spherical trigonometry. Al-Jayyani (989-1079), an Arabic mathematician in Islamic Spain, wrote what some consider the first treatise on spherical trigonometry, circa 1060, entitled “*The Book of Unknown Arcs of a Sphere*” in which spherical trigonometry was brought into its modern form [3]. This treatise later had a strong influence on European mathematics.

In the 13th century, Nasir al-Din al Tusi (1201–74) and al-Battani, continued to develop spherical trigonometry. Tusi was the first (c. 1250) to write a work on trigonometry independently of astronomy. The final major development in classical trigonometry was the invention of logarithms by the Scottish mathematician John Napier in 1614 that greatly facilitated the art of numerical computation—including the compilation of trigonometry tables [4].

III. NAVIGATIONAL TERMINOLOGY

Although the Earth is very round, in fact, it is a flattened sphere or spheroid with values for the radius of curvature of 6336 km at the equator and 6399 km at the poles. Approximating the earth as a sphere with a radius of 6370 km results in an actual error of up to about 0.5% [5]. The flattening of the ellipsoid is $\sim 1/300$ ($1/298.257222101$ is the defined value for the GPS ellipsoid WGS-84). *Flattening* is $(a-b)/a$ where a is the semi-major axis and b is the semi-minor axis. The value of a is taken as 6378.137 km in GPS ellipsoid WGS-84.

The position of a point on the surface of the Earth, or any other planet, for that matter, can be specified with two angles, latitude and longitude. These angles can be specified in degrees or radians. Degrees are far more common in geographic usage while radians win out during the calculation.

Latitude is the angle at the center of the Earth between the plane of the Equator (0° latitude) and a line through the center passing through the surface at the point in question. Latitude is positive in the Northern Hemisphere, reaching a limit of $+90^\circ$ at the North Pole, and negative in the Southern Hemisphere, reaching a limit of -90° at the South Pole. Lines of constant latitude are called *parallels*. *Longitude* is the angle at the center of the planet between two planes passing through the center and perpendicular to the plane of the Equator. One plane passes through the surface point in question, and the other plane is the *prime meridian* (0° longitude), which is defined by the location of the Royal Observatory in Greenwich, England. Lines of constant longitude are called *meridians*. All meridians converge at the north and south poles (90°N and -90°S). Longitudes typically range from -180° to $+180^\circ$. Longitudes can also be specified as east of Greenwich (E or positive) and west of Greenwich (W or negative). Fig. 1 illustrates the typical latitudes and longitudes on Earth. In spherical or geodetic coordinates, a *position* is a latitude taken

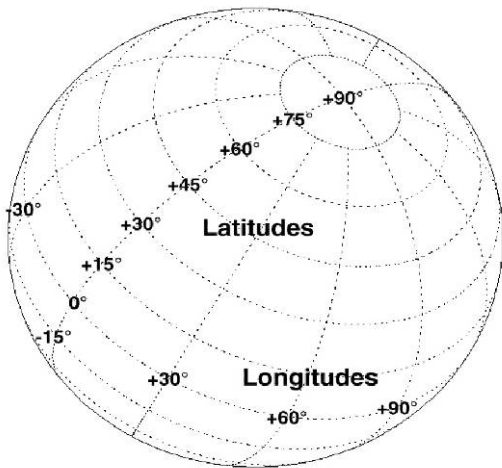


Fig. 1. Typical latitude and longitude values on earth surface.

together with a longitude, e.g., (lat,lon), which defines the horizontal coordinates of a point on the surface of a planet. *Azimuth* or *bearing* or *true course* is the angle a line makes with a meridian, taken clockwise from north. Usually azimuth is measured clockwise from north (0 = North, 90 = East, 180 = South 270= West, 360=0=North).

A *rhumb line* is a curve that crosses each meridian at the same angle. Although a great circle is a shortest path, it is difficult to navigate because your bearing (or azimuth) continuously changes as you proceed. Following a rhumb line covers more distance than following a geodesic, but it is easier to navigate. Unlike a great circle which encircles the earth, a pilot flying a rhumb line would spiral indefinitely poleward. The rumb line formulas are more complicated and will not be discussed.

IV. SPHERICAL TRIGONOMETRY

A. Great/Small Circles and Geodesic

Any plane will cut a sphere in a circle. A *great circle* is a section of a sphere by a plane passing through the center. Other circles are called *small circles*. All meridians are great circles, but all parallels, with the exception of the equator, are not. There is only one great circle through two arbitrary points that are not the opposite endpoints of a diameter. The smaller arc of the great circle through two given points is called a *geodesic*, and the length of this arc is the shortest distance on the sphere between the two points. The great circles on the sphere play a role similar to the role of straight lines on the plane.

B. Spherical Triangle

A figure formed by three great circle arcs pairwise connecting three arbitrary points on the sphere is called a *spherical triangle* or *Euler triangle* as shown in Fig. 2. The *vertices* of the triangles are formed by 3 vectors (OA, OB, OC). The angles less than π between the vectors are called the *sides* a, b and c of a spherical triangle. To each side of a

triangle there corresponds a great circle arc on the sphere. Each pair of vectors forms a plane. The *angles* A, B and C opposite the sides a, b and c of a spherical triangle are the angles between the great circle arcs corresponding to the sides of the triangle, or, equivalently, the angles between the planes determined by these vectors.

C. Spherical Triangle and Navigation

In navigation applications the angles and sides of spherical triangles have specific meanings. Sides (a, b or c) when multiplied by the radius of the Earth gives the geodesic distances between the points. By definition one *nautical mile* is equivalent to 1min of latitude extended at the surface of

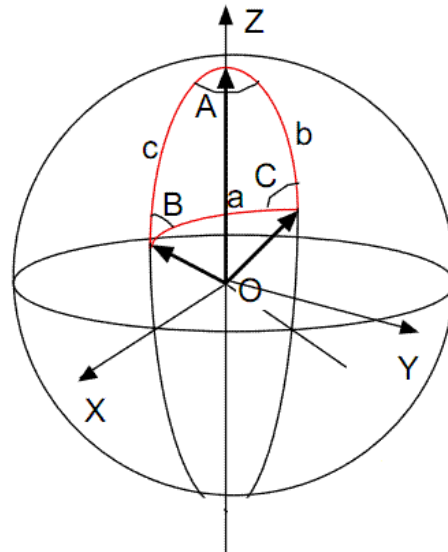


Fig. 2. A spherical triangle on a sphere.

Earth. When one point is the North pole, the two sides originating from that point (b and c in Fig. 2) are the *co-latitudes* of the other two points. The angles at the other two points (B and C in Fig. 2) are the *azimuth* or *bearing* to the other point.

D. Spherical Triangle Formulas

Most formulas from plane trigonometry have an analogous representation in spherical trigonometry. For example, there is a spherical law of sines and a spherical law of cosines. Let the sphere in Fig. 2 be a unit sphere. Then vectors OA, OB and OC are unit vectors. We take OA as the Z-axis, and OB projected into the plane perpendicular to OA as the X axis. Vectors OB and OC has components $(\sin c, 0, \cos c)$ and $(\sin b \cos A, \sin b \sin A, \cos b)$ respectively. From dot product rule:

$$\cos a = \overrightarrow{OB} \cdot \overrightarrow{OC}$$

$$\cos a = (\sin c, 0, \cos c) \cdot (\sin b \cos A, \sin b \sin A, \cos b)$$

This gives the identity (and its two analogous formulas) known as *law of cosines for sides*.

$$\cos a = \cos b \cos c + \sin b \sin c \cos A \quad (1)$$

$$\cos b = \cos c \cos a + \sin c \sin a \cos B \quad (2)$$

$$\cos c = \cos a \cos b + \sin a \sin b \cos C \quad (3)$$

Similarly by using the sin formula for vector cross product we get the *law of sines*.

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c} \quad (4)$$

The *law of cosines for angles* is given by.

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a \quad (5)$$

$$\cos B = -\cos C \cos A + \sin C \sin A \cos b \quad (6)$$

$$\cos C = -\cos A \cos B + \sin A \sin B \cos c \quad (7)$$

The proof can be found in [6] pp. 24. There are numerous other identities. A compact list can be found at [7]. All these identities allow us to solve the spherical triangles when appropriate angles and sides are given. See [6], [8] and [9].

The sum of the angles of a spherical triangle is between π and 3π radians (180° and 540°). The *spherical excess* is defined as $E = A + B + C - \pi$, and is measured in radians. The area A of spherical triangle with radius R and spherical excess E is given by the Girard's Theorem (8). A visual proof can be seen at [10].

$$A = R^2 E \quad (8)$$

The spherical geometry is a simplest model of elliptic geometry, which itself is a form of non-Euclidean geometry, where lines are geodesics. It is inconsistent with the "parallel line" postulate of Euclid. In the elliptic model, for any given line l and a point A , which is not on l , all lines through A will intersect l . Moreover the sum of angles in the triangle will be greater than 180° . For example for two of the sides, take lines of longitude that differ by 90° . For the third side, take the equator. This gives us a triangle with an angle sum of 270° .

V. NAVIGATIONAL CALCULATIONS

A. Distance and Bearing Calculation

The problem of determining the distance and bearing can easily be calculated. Let point B and C have positions ($lat1, lon1$) and ($lat2, lon2$) respectively. Let point A be the North Pole as shown in Fig. 2. The angle A is the difference between the longitudes. Moreover the sides b and c are ($90^\circ - lat1$) and ($90^\circ - lat2$) respectively. Keeping theses in mind and using (1) we get.

$$\cos a = \sin(lat1) \sin(lat2) + \dots \quad (9)$$

$$\cos(lat1) \cos(lat2) \cos(lon2 - lon1)$$

Taking \cos^{-1} we get the value of side a between 0 and π radians. By multiplying it with the radius of earth we get the required distance. The triangle can be solved for all sides. The angle B is the bearing from B to C. The values of $\sin B$ and $\cos B$ can be calculated using the flowing relations.

$$\sin B = \cos(lat2) \sin(lon2 - lon1)$$

$$\cos B = \cos(lat1) \sin(lat2) - \dots$$

$$\sin(lat1) \cos(lat2) \cos(lon2 - lon1)$$

The angle B can be computed using two-argument inverse tan function (usually denoted by $\tan 2^{-1}$ or atan2), which gives the value between π and $-\pi$ radians.

$$B = \tan 2^{-1}(\sin B, \cos B) \quad (10)$$

B. Dead Reckoning

Dead reckoning (DR) is the process of estimating one's current position based upon a previously determined position, or *fix*, and advancing that position based upon known speed, elapsed time (or distance) and course. In studies of animal navigation, dead reckoning is more commonly (though not exclusively) known as *path integration*, and animals use it to estimate their current location based on the movements they made since their last known location. Here we present the algorithm to compute the position of the destination if the distance and azimuth from previous position is known. Let the starting point has position (lat, lon). The azimuth from starting point is azm and the angular distance covered is dis in radians. Let the position of the destination be ($newlat, newlon$). Then latitude can be calculated by

$$newlat = \sin^{-1}(\sin(lat) \cos(dis) + \dots \cos(lat) \sin(dis) \cos(azm) \quad (11)$$

The calculation of longitude can be carried out as under.

$$lon1 = lon + \tan 2^{-1}(\sin(dis) \sin(azm), \dots$$

$$\cos(lat) \cos(dis) - \sin(lat) \sin(dis) \cos(azm)$$

The value of $lon1$ can be outside the rang of $-\pi$ and π radians. The function angpi2pi brings it in the required range.

$$newlon = \text{angpi2pi}(lon1) \quad (12)$$

C. Software Implementation

The above formulas can easily be implemented in any high level language. A good resource for the above formulas is [11]. Please note that it uses opposite longitude convention. It also contains links to Javascript calculator with elliptical earth models and Microsoft Excel implementation. A Visual Basic implantation can be found in [12]. A Python implementation can be found at [13] and [14]. The Mapping Toolbox in MATLAB has a wealth of function for navigation, mapping and GIS. The main functions are *distance*, *reckon* and *dreckon*. The source code for most of the functions can be read from the corresponding m-files. See the local help available from Help menu. The online documentation for latest Mapping Toolbox is available from [15].

VI. CONCLUSION

Spherical trigonometry is used for most calculations in navigation and astronomy. For the most accurate navigation and map projection calculation, ellipsoidal forms of the equations are used but these equations are much more complex. Dead reckoning is used extensively in Inertial Navigation Systems (INS). Spherical trigonometry along with linear algebra forms the backbone for modern navigation systems such as GPS. It is a prerequisite for good

understanding of GIS. It is much more pertinent to integrate course of spherical trigonometry in the engineering curriculum.

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